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MINIMUM-WEIGHT DESIGN
FOR MOVING LOADS

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PREPARED FOR:

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in brief

This report derives a mathematical model that describes the minimum-weight design of a horizontal beam under an invariable, concentrated, vertical load moving slowly from one end to the other. Beams of this type, although possibly impractical in earthbound structures, may someday become interesting for applications in a weightless environment such as that of a space station.

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PREFACE

Plastic design of structures has received considerable
attention in recent years. To the authors' knowledge, the
present Project RAND memorandum is an innovation in this
field and also in the domain of linear programming — in the
former because the beam design is to withstand a one-parameter
family of load conditions rather than a single load distribu-
tion, and in the latter because the program involves a continuum
rather than a finite number of variables.

The basic mathematical treatment of the continuous version
of the problems presented in this paper is contained in RM-2993 PR,
"A Linear Program of Prager's: Notes on Linear Programming and
Extensions — Part 60," now in the process of publication. In
the present paper, which is to be submitted for presentation at
the Fourth U. S. National Congress of Applied Mechanics at
Berkeley, California, June, 1962, the mathematical model is derived
from physical considerations and solved in a form readily accessi-
ble to the structural engineer.

It might be added that beams of the type considered in this
paper may be impractical (i.e., expensive to manufacture) for
earth-bound structures. They are, however, important for theo-
retical reasons since they represent, in a sense, the ultimate
in weight design. It is of course conceivable that such designs
might become more important in the future as actual components
of nonearth-bound structures such as space stations.

SUMMARY

The main problem considered and solved ~~in this paper~~
is the minimum—weight design of a horizontal I—beam of con—
stant web and variable flange thickness. The beam is simply
supported at one end and built in at the other and is designed
to withstand (in plastic flow) a concentrated vertical load
of fixed intensity moving slowly from one end to the other.

To introduce ~~the reader to~~ the general techniques
employed, a few discrete versions of the problem are first
presented and solved as linear—programming problems.

CONTENTS

MINIMUM—WEIGHT DESIGN FOR MOVING LOADS

1. INTRODUCTION

Plastic design of continuous beams and frames for mini—
mum weight has been extensively studied in recent years.
Almost without exception, earlier studies dealt with structures
of given over—all dimensions (widths of bays and heights of
stories) that are subjected to a single system of concentrated
loads judged to be the most dangerous combination of the
groups of loads that the structure has to carry. For practi—
cal reasons, the beams and columns of steel frames are
usually prismatic, so that the designer disposes of only a
finite number of design parameters (the yield moments of the
beams and columns). If, within the range of practical
structural sections, the unit weight of a section can be
approximated by a linear function of its yield moment,
designing for minimum weight can then be formulated as a
problem in linear programming (see, for instance, [1]*, p. 84
ff.).

While designs with beams or columns of continuously
varying cross section would not, in general, be practical,
they are worth investigating because they furnish the
theoretical minimum of the structural weight, against which
practical designs may be checked.

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*Numbers in brackets indicate items in the list of
references at the end of the paper.

The minimum-weight design of a beam with continuously
varying cross section subjected to a single system of loads
has been discussed by Heyman [2]. Consider, for instance,
a sandwich beam with a core of given constant height h and
breadth b, and flanges of constant breadth b and variable
thickness t. Since the designer controls only the thickness
t, the variable part of the structural weight is proportional
to the integral of t extended along the beam. Using a theorem
of Drucker and Shield [3], Heyman has shown that the abso-
lute minimum-weight design of such a beam can be based on a
deflected shape for which the curvature has constant abso-
lute value. Figure 1 shows deflections of this kind for a
beam that is simply supported at one end $(x = 0)$ and built
in at the other end $(x = \mathit{l})$. The curvature is found to
change sign at $x = \mathit{l}/\sqrt{2}$. Accordingly, the bending moment
must vanish at $x = \mathit{l}/\sqrt{2}$. Since the indeterminacy of the
beam is removed by this condition, the bending moments can be
obtained from the design loads without reference to the cross-
sectional dimensions of the beam. The flange thickness at
each cross section is then determined from the bending
moment at this section in such a way that the yield limit of
the flange material is reached at all sections when the beam
is subjected to the design loads.

As this example shows, an absolute minimum-weight design
for a single system of loads is essentially a statically

determinate design. This feature, which greatly facilitates
the design, is likely to be absent when a structure has to
be designed for more than one system of loads. To illustrate
the arguments that arise in the solution of problems of this
kind, the sandwich beam in Fig. 1 will be designed for a
moving load of constant intensity P that may act at any
section of the beam. Calling for the determination of the
flange thickness at each station x, this problem involves
an infinity of design parameters or, what amounts to the
same, a design function $t(x)$. Since there are as yet no
general methods of solving continuous problems of this kind,
a discrete analog will be discussed in Sec. 2 as a guide
to the solution of the continuous problem, which will be
given in Sec. 3.

2. DISCRETE POSITIONS OF THE LOAD

The general features of the desired design can be
explored in the following way. Dividing the span ℓ of the
beam into n equal segments, suppose that only the partition
points are eligible as points of application of the load, and
that the flange thickness t is a segmentwise linear function
of the distance x measured along the beam. This kind of
design is specified by the n values of t at the points $x = \ell/n$,
$2\ell/n, \ldots, (n-1)\ell/n, \ell$, the flange thickness being zero at
the simply supported end $x = 0$. It is to be expected that
the desired design with continuously varying flange thickness

results as $n \longrightarrow \infty$.

If h denotes the constant height and b the constant breadth of the core, and s_o the tensile or compressive yield stress of the flange, and if the flange thickness t is treated as small in comparison to h, the yield moment is

$$(2.1) \qquad Y = s_o \, bht.$$

For a unit length of the beam, the variable part of the structural weight, that is, the weight of the flanges, is proportional to the flange thickness t and hence to the yield moment (2.1).

For $n = 2$, a design of the considered type is specified by the yield moments Y_1 and Y_2 at the center of the beam and the built—in end. Only one position of the load and one yield mechanism need to be investigated. With the deflections and hinge rotations shown in Fig. 2, the condition that the energy dissipated in the plastic hinges equals the work of the load takes the form

$$(2.2) \qquad 2Y_1 + Y_2 = P\ell/2.$$

Since the yield moment vanishes at the simply supported end, the average values of the limiting moment in the two halves of the span are $Y_1/2$ and $(Y_1 + Y_2)/2$, so that the variable part of the structural weight is proportional to

$$(2.3) \qquad W = (2\,Y_1 + Y_2)\ell/4.$$

The yield moments Y_1 and Y_2 of a minimum—weight design must satisfy (2.2). Substitution of this equation into (2.3) shows the variable part of the structural weight to be proportional to

$$(2.4) \qquad W = P\ell^2/8.$$

Thus there exist infinitely many minimum—weight designs for $n = 2$.

For a greater value of n, a yield mechanism is defined by the locations of its two plastic hinges, say $x = h\ell/n$ and $x = k\ell/n$ with $k > h$. From the deflections and hinge rotations indicated in Fig. 3, the condition that the energy dissipation in the plastic hinges must not exceed the work of the load becomes

$$(2.5) \qquad kY_h + hY_k \geq Ph(k-h)\ell/n.$$

The variable part of the structural weight is proportional to

$$(2.6) \qquad W = (Y_1 + Y_2 + \ldots + Y_{n-1} + 0.5\,Y_n)\ell/n.$$

For $n = 4$, for instance, the (nonnegative) yield moments of the minimum—weight design must minimize the linear form
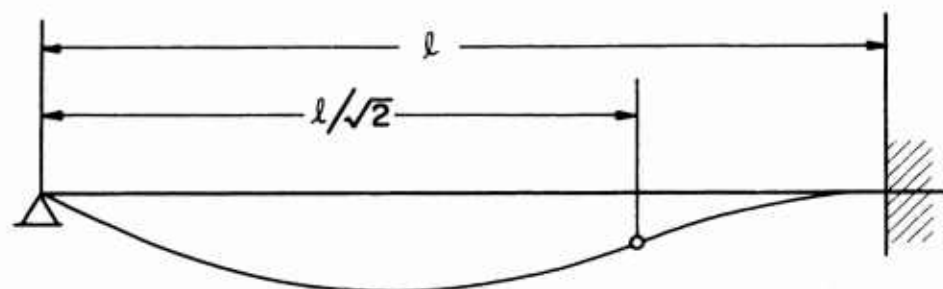
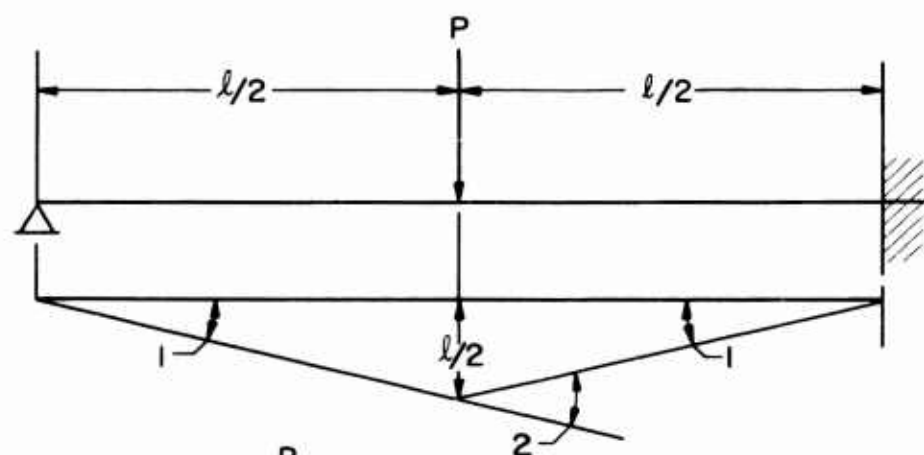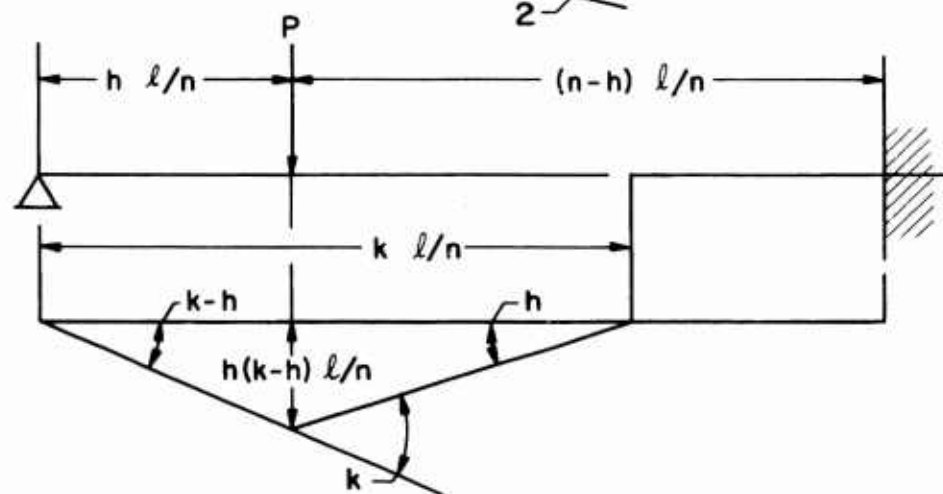$$(2.7) \qquad W = (Y_1 + Y_2 + Y_3 + 0.5\,Y_4)\ell/4,$$

FIG. 1

FIG. 2

FIG. 3

while satisfying the inequalities

$$(2.8) \qquad 2Y_1 + Y_2 \qquad\qquad\qquad \geq P\ell/4,$$
$$3Y_1 \qquad + Y_3 \qquad\qquad \geq P\ell/2,$$
$$4Y_1 \qquad\qquad + Y_4 \quad \geq 3P\ell/4,$$
$$3Y_2 + 2Y_3 \qquad\qquad \geq P\ell/2,$$
$$4Y_2 \qquad + 2Y_4 \quad \geq P\ell,$$
$$4Y_3 + 3Y_4 \geq 3P\ell/4,$$

which are obtained from (2.5) by letting h and k vary from 1 to 4 subject to the condition that $k > h$.

In general, the n yield moments of the minimum-weight design must minimize a linear form, which represents the variable part of the structural weight, while satisfying $n(n-1)/2$ inequality constraints. Expressed in structural terms, the essential features of the solution of this linear-programming problem are as follows (see, for instance, [4]). The yield moments of the minimum-weight design fulfill n of the $n(n-1)/2$ constraints as equations rather than inequalities. Each of these equations corresponds to a yield mechanism that becomes critical for some position of the moving load; it states that the energy dissipated in the plastic hinges of this mechanism must equal the work of the load if the mechanism is to be critical. The n equations can be linearly combined with nonnegative factors in such a manner that the coefficient of each yield moment in the

linear combination equals the coefficient of this yield
moment in the expression for the variable part of the
structural weight.

For $n = 4$, the use of the equality sign in the second,
fourth, fifth, and sixth relations (2.8) furnishes the yield
moments

(2.9) $Y_1 = 5P\ell/32, \; Y_2 = 7P\ell/48, \; Y_3 = P\ell/32, \; Y_4 = 5P\ell/24,$

which are readily seen to satisfy the remaining two relations
(2.8). Moreover, when the four equations corresponding to
critical yield mechanisms are combined with the factors $\ell/12$,
$\ell/24$, $\ell/32$, $\ell/48$, the coefficient of each yield moment in
the linear combination is found to equal the coefficient of
this yield moment in (2.7). The yield moments (2.9) there—
fore represent the minimum—weight design.

In Fig. 4, the line OABCD represents the variation of
the yield moment along the beam; and OA'B'C'D' is the image
of this line, by symmetry with respect to the x—axis. For
the considered positions of the load at $x = \ell/4$, $x = \ell/2$,
and $x = 3\ell/4$, the bending moments are represented by the
lines OAC'E, OBC'D', and OCD', respectively. For none of
these load positions does the absolute value of the bending
moment at any section of the beam exceed the yield moment
at this section, while for each load position the absolute
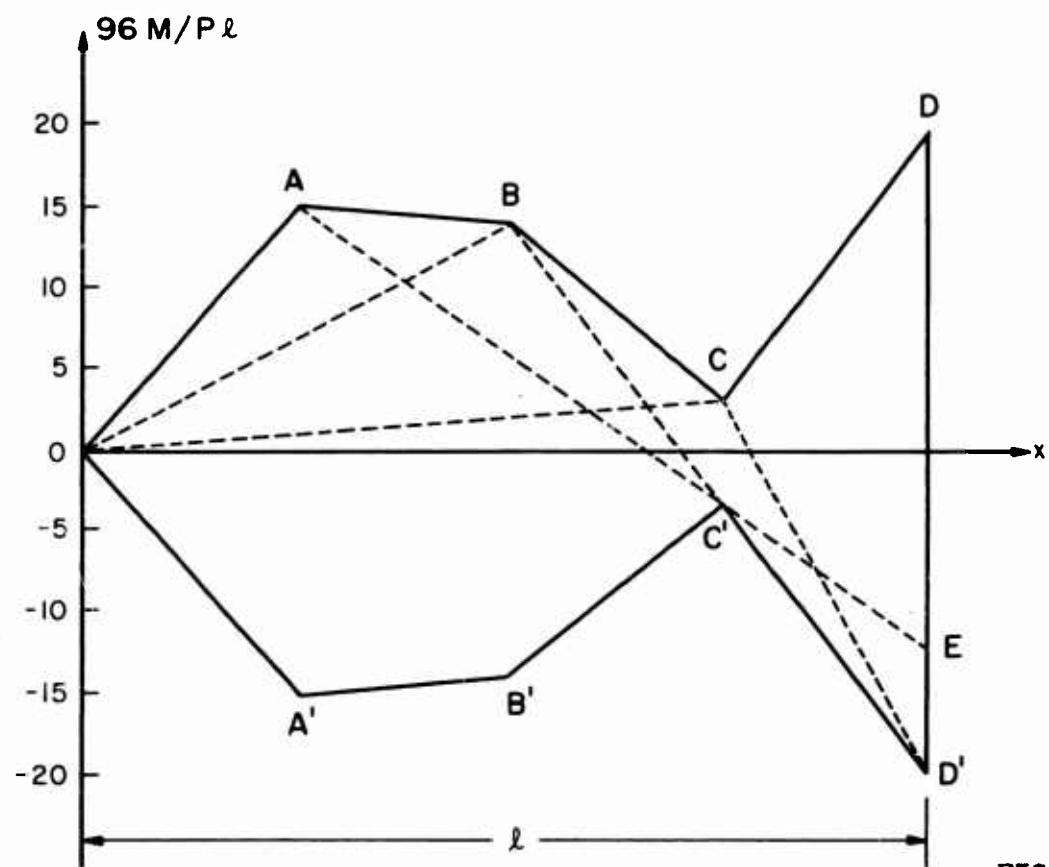value of the bending moment attains the yield moment in at

FIG. 4

least two sections.

With the yield moments (2.9), (2.7) furnishes

(2.10) $W = 7P\ell^2/64$.

3. CONTINUOUSLY VARYING POSITION OF THE LOAD

The solution of the discrete problem in Sec. 2 suggests that for continuously varying position of the load the yield moments of the minimum weight design are represented by arcs such as OABC and CD in Fig. 5. Denoting the abscissa of C by x_1, suppose that the load acts at the section $x = x_0 \leq x_1$ and that a plastic hinge forms at this section. The bending moment at this section then equals the yield moment $Y(x_0)$, and the reaction at the simply supported end equals

(3.1) $R(x_0) = Y(x_0)/x_0$.

To the left of the load, the bending moments are given by

(3.2) $M(x;x_0) = x\,R(x_0) = x\,Y(x_0)/x_0$, $0 \leq x \leq x_0$

(line OB in Fig. 5). If the arc OABC is concave with respect to the x-axis, these bending moments nowhere exceed the yield moment. To the right of the load, the bending moments are given by

(3.3) $M(x;x_0) = x\,R(x_0) - (x - x_0)P$, $x_0 \leq x \leq \ell$

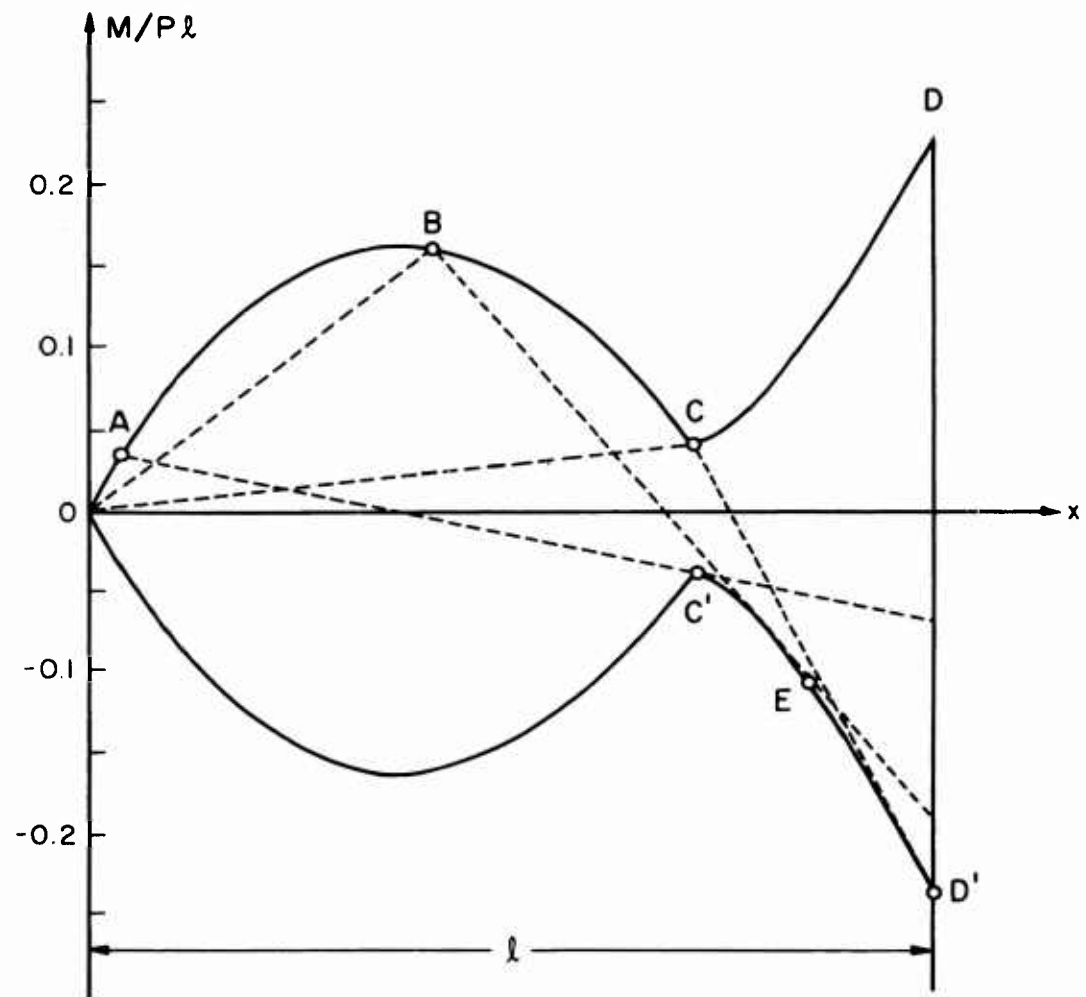(line BE in Fig. 5). Since there must be a yield mechanism

FIG. 5

corresponding to the considered position of the load, the absolute value of the bending moment must attain the yield moment at some abscissa $x = x_2 \geq x_1$ without exceeding the yield moment at any other abscissa. In other words, the line BE must be tangent to the image C'D' of the arc CD with respect to the x-axis. Thus, one obtains

$$(3.4) \qquad Y(x) + M(x; x_0) \geq 0 \quad , \quad x_1 \leq x \leq \ell,$$

with equality for one value of x. With the use of (3.1) and (3.3), one finds

$$(3.5) \qquad Y(x) \geq (x - x_0)P - x\,Y(x_0)/x_0 \;, \quad x_1 \leq x \leq \ell.$$

To find the abscissa of the point of contact E of the line BE and the arc C'D', equate the derivative of the right-hand side of (3.5) to zero and replace x by x_2. Thus, x_2 is given by

$$(3.6) \qquad x_2 = -\frac{P}{[Y(x_0)/x_0]'} \quad , \quad 0 \leq x_0 \leq x_1 \;,$$

where the prime denotes differentiation with respect to x_0. With this value of x_2, the yield moment $Y(x_2)$ is found by setting $x = x_2$ and using the equality sign in (3.5). As x_0 varies from 0 to x_1, the point of contact E moves from C' to D' as indicated in Fig. 5. Accordingly, the arc C'D' is uniquely determined by the arc OABC.

The variable part of the structural weight is propor-

tional to

$$(3.7) \qquad W = \int_0^{x_1} Y(x) \, dx + \int_{x_1}^{\ell} Y(x) \, dx \, .$$

Let $Y(x_o)$ be an arbitrarily defined positive concave function in $0 \leq x_o \leq x_1$ that vanishes for $x_o = 0$, and $x = f(x_o)$ an arbitrary increasing function that maps the interval $(0, x_1)$ onto (x_1, ℓ). When the second integral in (3.7) is transformed by the use of the inequality (3.5) and the change of variable $x = f(x_o)$, the relation

$$(3.8) \qquad W \geq \int_0^{x_1} \left[Y(x_o) \left(1 - \frac{ff'}{x_o} \right) + (f - x_o) f' P \right] dx_o$$

is obtained. To render the lower bound (3.8) for the structural weight independent of the choice of $Y(x_o)$, set

$$(3.9) \qquad ff' = x_o,$$

or

$$(3.10) \qquad f = \sqrt{x_o^2 + c^2} \, ,$$

where c is a constant of integration.

Since f is to map the interval $(0, x_1)$ onto (x_1, ℓ), f satisfies

$$(3.11) \qquad f(0) = x_1 \, , \quad f(x_1) = \ell \, .$$

It follows from (3.10) and (3.11) that

$$(3.12) \qquad c = x_1 = \ell / \sqrt{2} \, .$$

and

$$(3.13) \qquad f = \sqrt{x_o^2 + x_1^2} \; .$$

In view of (3.9) and (3.13), the inequality (3.8) reduces to

$$(3.14) \qquad W \geq \tfrac{1}{4}(1 - \sqrt{2} + \log(1 + \sqrt{2}))P\ell^2 = 0.1168P\ell^2.$$

To obtain the function $Y(x_o)$ for which W equals this lower bound, identify the function (3.14) with the right-hand side of (3.6) to obtain

$$(3.15) \qquad (\frac{Y(x_o)}{x_o})' = -\frac{P}{\sqrt{x_o^2 + x_1^2}} \; ,$$

or

$$(3.16) \qquad Y(x_o) = Px_o \left\{ k - \log \left[(x_o + \sqrt{x_o^2 + x_1^2})/\ell \right] \right\}, \; 0 \leq x_o \leq x_1,$$

where k is a constant of integration. To determine this constant, note that $Y(x_1)$ evaluated from (3.16) must equal $Y(x_1)$ obtained by substituting $x_o = 0$ and $x = x_1$ into (3.5) and using the equality sign in this relation. Thus, the value of k is

$$k = \tfrac{1}{2}(1 + \log \frac{1 + \sqrt{2}}{2}).$$

With $f = x_2$, it follows from (3.13) that

$$(3.18) \qquad x_o = \sqrt{x_2^2 - x_1^2},$$

and from (3.5) and (3.16) that

$$(3.19)$$
$$Y(x_2) = P \left\{ x_2 \sqrt{x_2^2 - x_1^2} - x_2 \sqrt{x_2^2 - x_1^2} \; k \right.$$
$$\left. - \log[(x_2 + \sqrt{x_2^2 - x_1^2}/\ell] \right\}, \; x_1 \leq x_2 \leq \ell,$$

where k is given by (3.17). The arcs OABC and CD in Fig. 5 represent the functions (3.16) and (3.19).

It is interesting to note that the abscissa x_1 defined by (3.12) is identical with the abscissa at which the curvature in Fig. 1 changes sign. An analogous identity was found to exist for the doubly built–in beam, which was treated along similar lines, but no general reason for this has been dis–covered.

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